TESTING QUATERNION PROPERNESS: GENERALIZED LIKELIHOOD RATIOS AND LOCALLY MOST POWERFUL INVARIANTS

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ABSTRACT

This paper considers the problem of determining whether a quaternion random vector is proper or not, which is an important problem because the structure of the optimal linear processing depends on the specific kind of properness. In particular, we focus on the Gaussian case and consider the two main kinds of quaternion properness, which yields three different binary hypothesis testing problems. The testing problems are solved by means of the generalized likelihood ratio tests (GLRTs) and the locally most powerful invariant tests (LMPITs), which can be derived even without requiring an explicit expression for the maximal invariant statistics. Some simulation examples illustrate the performance of the proposed tests, which allows us to conclude that, for moderate sample sizes, it is advisable to use the LMPITs.

1. INTRODUCTION

During the last years, quaternion signal processing has received increasing interest due to its applications in problems such as design of space-time block codes [1], analysis of polarized waves [2], or modeling of wind profiles [3]. The application-oriented research has also been complemented with some theoretical works, such as those considering the statistical characterization of quaternion random vectors [4], where it has been proved that the structure of the optimal linear processing depends on the particular kind of properness, and therefore it becomes crucial to determine whether our quaternion data are proper or not.

This paper considers the problem of testing for the properness of a quaternion Gaussian vector. Thus, we revisit a recent derivation of three generalized likelihood ratio tests (GLRTs) [5, 6], and complement this work with their locally most powerful invariant counterparts. Unlike the GLRTs, the derivation of the locally most powerful invariant tests (LMPITs) is rather involved. However, thanks to the Wijsman's theorem [7], and by correctly exploiting the specific invariances of each testing problem, one can obtain the LMPIT statistics without the need of an explicit expression for the maximal invariants. In particular, the GLRT and LMPIT statistics for our testing problems are respectively given by the determinant and Frobenius norm of the corresponding sample coherence matrices. Additionally, we point out several interesting connections with the problems of testing for the properness of a complex random vector [8,9], and with the sphericity tests for real and complex random vectors [10, 11]. Finally, the advantage of the LMPITs over the GLRTs is illustrated by means of some numerical examples.

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2. PRELIMINARIES

We will use bold-faced upper case letters to denote matrices, bold-faced lower case letters for column vectors, and light-faced lower case letters for scalar quantities. Superscripts $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ denote quaternion (or complex) conjugate, transpose and Hermitian (i.e., transpose and quaternion conjugate), respectively. The notation $\mathbf{A} \in \mathbb{F}^{m\times n}$ denotes that $\mathbf{A}$ is a $m \times n$ matrix with entries in $\mathbb{F}$, where $\mathbb{F}$ can be $\mathbb{R}$, the field of real numbers, $\mathbb{C}$, the field of complex numbers, or $\mathbb{H}$, the skew-field of quaternion numbers. $\Re(\mathbf{A})$, $\mathrm{Tr}(\mathbf{A})$, $\|\mathbf{A}\|$, and $|\mathbf{A}|$ denote the real part, trace, Frobenius norm, and determinant of matrix $\mathbf{A}$. $\mathbf{A}^{1/2}$ (respectively $\mathbf{A}^{-1/2}$) is the Hermitian square root of the Hermitian matrix $\mathbf{A}$ (resp. $\mathbf{A}^{-1}$). The diagonal matrix with vector $\mathbf{a}$ along its diagonal is denoted by $\mathrm{diag}(\mathbf{a})$, $\mathbf{I}_n$ is the identity matrix of dimension n, and $\mathbf{0}_{m\times n}$ is the $m \times n$ zero matrix. Finally, the Kronecker product is denoted by $\otimes$, E is the expectation operator, and in general $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is the cross-correlation matrix for vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., $\mathbf{R}_{\mathbf{a},\mathbf{b}} = E\mathbf{a}\mathbf{b}^H$.

2.1 Quaternion Algebra

Quaternions are hypercomplex numbers defined by

$$x = r_1 + \eta r_\eta + \eta' r_{\eta'} + \eta'' r_{\eta''},$$

where $r_1, r_\eta, r_{\eta'}, r_{\eta''} \in \mathbb{R}$ are four real numbers, and the three imaginary units[1] (η, η', η'') satisfy

$$\eta^2 = \eta'^2 = \eta''^2 = \eta\eta'\eta'' = -1,$$

which also implies $\eta\eta' = \eta''$, $\eta'\eta'' = \eta$, and $\eta''\eta = \eta'$.

Quaternions form a skew field $\mathbb{H}$ [12], and therefore they satisfy the axioms of a field except the commutative law of the product, i.e., for $x, y \in \mathbb{H}$, $xy \neq yx$ in general. The conjugate of a quaternion x is $x^* = r_1 - \eta r_\eta - \eta' r_{\eta'} - \eta'' r_{\eta''}$, and the inner product of two quaternions x, y is defined as xy^*. Two quaternions are orthogonal if and only if (iff) their scalar product (the real part of the inner product) is zero, and the norm of a quaternion x is $|x| = \sqrt{xx^*} = \sqrt{r_1^2 + r_\eta^2 + r_{\eta'}^2 + r_{\eta''}^2}$. Furthermore, we say that ν is a pure unit quaternion iff $\nu^2 = -1$ (i.e., iff $|\nu| = 1$ and its real part is zero).

Quaternions also admit the Euler representation

$$x = |x|e^{\nu\theta} = |x|\left(\cos\theta + \nu\sin\theta\right),$$

[1] In this paper we use the general representation $\{\eta, \eta', \eta''\}$ instead of the conventional canonical basis $\{i, j, k\}$.

where ν is a pure unit quaternion and $\theta \in \mathbb{R}$ is the angle (or argument) of the quaternion. Taking this into account we can easily define the rotation and involution operations [12]:

Definition 1 (Rotation and Involution) *Consider a non-zero quaternion $a = |a|e^{\nu\theta} = |a|(\cos\theta + \nu\sin\theta)$, then*

$$x^{(a)} = axa^{-1},$$

represents a three-dimensional rotation of the imaginary part of x. Specifically, the vector $[r_\eta, r_{\eta'}, r_{\eta''}]^T$ is rotated an angle 2θ in the pure imaginary plane orthogonal to ν. In the particular case of pure quaternions ν, $x^{(\nu)}$ represents a rotation of angle π, which is an involution.

Finally, a quaternion x can also be represented by means of the *Cayley-Dickson* construction $x = a_1 + \eta'' a_2$, where

$$a_1 = r_1 + \eta r_\eta, \qquad a_2 = r_{\eta''} + \eta r_{\eta'},$$

can be seen as complex numbers in the plane $\{1, \eta\}$.

2.2 Second-Order Statistics of Quaternion Vectors

The second-order statistics (SOS) of a n-dimensional quaternion random vector $\mathbf{x} = \mathbf{r}_1 + \eta\mathbf{r}_\eta + \eta'\mathbf{r}_{\eta'} + \eta''\mathbf{r}_{\eta''}$ are obviously given by the joint SOS of the vectors $\mathbf{r}_1, \mathbf{r}_\eta, \mathbf{r}_{\eta'}, \mathbf{r}_{\eta''} \in \mathbb{R}^{n\times 1}$ in its real representation. However, analogously to the case of complex random vectors [13], the statistical analysis can benefit from the definition of an augmented quaternion vector[2] $\bar{\mathbf{x}} = \left[\mathbf{x}^T, \mathbf{x}^{(\eta)^T}, \mathbf{x}^{(\eta')^T}, \mathbf{x}^{(\eta'')^T}\right]^T$. Thus, the SOS of $\mathbf{x}$ are given by the augmented covariance matrix [4]

$$\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = \begin{bmatrix} \mathbf{R}_{\mathbf{x},\mathbf{x}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}} \\ \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta')}} \\ \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta)}} \\ \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix},$$

which contains the covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}} = E\mathbf{x}\mathbf{x}^H$ and three complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} = E\mathbf{x}\mathbf{x}^{(\eta)^H}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} = E\mathbf{x}\mathbf{x}^{(\eta')^H}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}} = E\mathbf{x}\mathbf{x}^{(\eta'')^H}$. Interestingly, this representation allows us to easily relate the SOS of the quaternion vector $\mathbf{x}$ and those of some common transformations [4]:

Lemma 1 *Consider the* full-widely *linear transformation*

$$\mathbf{y} = \mathbf{F}_{\bar{\mathbf{x}}}^H\bar{\mathbf{x}} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_\eta^H\mathbf{x}^{(\eta)} + \mathbf{F}_{\eta'}^H\mathbf{x}^{(\eta')} + \mathbf{F}_{\eta''}^H\mathbf{x}^{(\eta'')},$$

where $\mathbf{F}_{\bar{\mathbf{x}}} = \left[\mathbf{F}_1^T, \mathbf{F}_\eta^T, \mathbf{F}_{\eta'}^T, \mathbf{F}_{\eta''}^T\right]^T \in \mathbb{H}^{4n\times n}$. Then, the SOS of $\mathbf{y}$ are given by $\mathbf{R}_{\bar{\mathbf{y}},\bar{\mathbf{y}}} = \overline{\mathbf{F}}^H\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\overline{\mathbf{F}}$, where

$$\overline{\mathbf{F}} = \underbrace{\begin{bmatrix} \mathbf{F}_1 & \mathbf{F}^{(\eta)}_\eta & \mathbf{F}^{(\eta')}_{\eta'} & \mathbf{F}^{(\eta'')}_{\eta''} \\ \mathbf{F}_\eta & \mathbf{F}^{(\eta)}_1 & \mathbf{F}^{(\eta')}_{\eta''} & \mathbf{F}^{(\eta'')}_{\eta'} \\ \mathbf{F}_{\eta'} & \mathbf{F}^{(\eta)}_{\eta''} & \mathbf{F}^{(\eta')}_1 & \mathbf{F}^{(\eta'')}_\eta \\ \mathbf{F}_{\eta''} & \mathbf{F}^{(\eta)}_{\eta'} & \mathbf{F}^{(\eta')}_\eta & \mathbf{F}^{(\eta'')}_1 \end{bmatrix}}_{4n\times 4n}.$$

[2]From now on, we will use the notation $\mathbf{F}^{(a)}$ to denote the element-wise rotation of matrix $\mathbf{F}$.

Lemma 2 *A rotation $\mathbf{y} = \mathbf{x}^{(a)}$ results in a simultaneous rotation of the orthogonal basis $\{1, \eta, \eta', \eta''\}$ and the augmented covariance matrix*

$$\mathbf{R}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}(\{1, \eta, \eta', \eta''\}) = \mathbf{R}^{(a)}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\{1, \eta^{(a^*)}, \eta'^{(a^*)}, \eta''^{(a^*)}\}),$$

where the expressions in parentheses make explicit the bases for the augmented covariance matrices.

Lemma 3 *The augmented covariance matrices in two different orthogonal bases are related as*

$$\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\{1, \nu, \nu', \nu''\}) = \mathbf{\Gamma}\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\{1, \eta, \eta', \eta''\})\mathbf{\Gamma}^H,$$

where

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{\Lambda}_\nu\mathbf{Q}\mathbf{\Lambda}_\eta^H \end{bmatrix} \otimes \mathbf{I}_n,$$

$\mathbf{Q} \in \mathbb{R}^{3\times 3}$ is the rotation matrix for the change of basis $[\nu, \nu', \nu''] = [\eta, \eta', \eta'']\mathbf{Q}^T$, $\mathbf{\Lambda}_\nu = diag\left([\nu, \nu', \nu'']^T\right)$, and $\mathbf{\Lambda}_\eta = diag\left([\eta, \eta', \eta'']^T\right)$.

2.3 Properness of Quaternion Random Vectors

Analogously to the complex case [13], the structure of the optimal linear processing of quaternion random vectors depends on the quaternion properness. In [4] (see also the references therein), the authors have presented two main kinds of quaternion properness:

Definition 2 ($\mathbb{Q}$-Properness) *A quaternion random vector $\mathbf{x}$ is $\mathbb{Q}$-proper iff the three complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ vanish.*

Definition 3 ($\mathbb{C}^\eta$-Properness) *A quaternion random vector $\mathbf{x}$ is $\mathbb{C}^\eta$-proper iff the complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ vanish.*

Here we must note that, as a direct consequence of Lemma 3, $\mathbb{Q}$-properness implies $\mathbb{C}^\eta$-properness for all η. Furthermore, the $\mathbb{C}^\eta$-properness definition is directly related to the complex properness of the vectors in the Cayley-Dickson representation of $\mathbf{x}$ [4]:

Lemma 4 *A quaternion random vector $\mathbf{x}$ is $\mathbb{C}^\eta$-proper iff the complex vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{C}^{n\times 1}$ in its Cayley-Dickson representation $\mathbf{x} = \mathbf{a}_1 + \eta''\mathbf{a}_2$ are jointly proper, i.e., iff the complex vector $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$ is proper ($\mathbf{R}_{\mathbf{a},\mathbf{a}^*} = \mathbf{0}_{2n\times 2n}$).*

From a practical point of view, the main implications of the properness definitions consist in the simplification of the optimal linear processing of quaternion random vectors. In the general case, the optimal linear processing is *full-widely* linear, i.e., we must simultaneously operate on the quaternion random vector and its three involutions. However, in the case of proper vectors the optimal linear processing simplifies as follows [4]:

Lemma 5 (Semi-widely linear processing) *The optimal linear processing of $\mathbb{C}^\eta$-proper vectors is semi-widely linear*

$$\mathbf{y} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_\eta^H\mathbf{x}^{(\eta)}.$$

Lemma 6 (Conventional linear processing) *The optimal linear processing of $\mathbb{Q}$-proper vectors takes the form*

$$\mathbf{y} = \mathbf{F}_1^H \mathbf{x},$$

i.e., we do not need to operate on the quaternion involutions.

Finally, in [4] the authors introduced a third kind of quaternion properness, which can be interpreted as the *difference* between $\mathbb{C}^\eta$ and $\mathbb{Q}$ properness.

Definition 4 ($\mathbb{R}^\eta$-Properness) *A quaternion random vector $\mathbf{x}$ is $\mathbb{R}^\eta$-proper iff the complementary covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$ vanishes.*

3. TESTING FOR PROPERNESS OF QUATERNION GAUSSIAN VECTORS

3.1 Problem Formulation

Analogously to the complex case [8,9], determining the kind of properness of a quaternion random vector is an important problem because it establishes the most convenient kind of linear processing. In this paper, we focus on quaternion Gaussian vectors and define the three following hypotheses:

- $\mathscr{H}_{\mathbb{Q}}$: The quaternion random vector $\mathbf{x}$ is $\mathbb{Q}$-proper.
- $\mathscr{H}_{\mathbb{C}^\eta}$: The quaternion random vector $\mathbf{x}$ is $\mathbb{C}^\eta$-proper.
- $\mathscr{H}_{\mathscr{I}}$: The quaternion random vector $\mathbf{x}$ is not constrained to be $\mathbb{Q}$-proper nor $\mathbb{C}^\eta$-proper.

Thus, we will consider three different testing problems: 1) the problem of determining whether $\mathbf{x}$ is $\mathbb{Q}$-proper or not ($\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathscr{I}}$); 2) the problem of determining whether $\mathbf{x}$ is $\mathbb{C}^\eta$-proper ($\mathscr{H}_{\mathbb{C}^\eta}$ versus $\mathscr{H}_{\mathscr{I}}$); 3) the problem of determining whether the $\mathbb{C}^\eta$-proper vector $\mathbf{x}$ is also $\mathbb{Q}$-proper ($\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathbb{C}^\eta}$).

3.2 Maximal Invariant Statistics

Before proceeding, let us summarize the invariances of the quaternion properness definitions:

- $\mathbb{Q}$-Properness: Taking into account Lemmas 1-3, it is easy to see that the $\mathbb{Q}$-properness definition is invariant to rotations and invertible *conventional* linear transformations, i.e., $\mathbf{x}$ is $\mathbb{Q}$-proper iff $\mathbf{y} = \mathbf{F}_1^H \mathbf{x}^{(a)}$ is $\mathbb{Q}$-proper for all non-null $a \in \mathbb{H}$ and invertible $\mathbf{F}_1 \in \mathbb{H}^{n\times n}$.
- $\mathbb{C}^\eta$-Properness: Analogously, the $\mathbb{C}^\eta$-properness definition is invariant to invertible *semi-widely* linear transformations, i.e., $\mathbf{x}$ is $\mathbb{C}^\eta$-proper iff $\mathbf{y} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_\eta^H \mathbf{x}^{(\eta)}$ is $\mathbb{C}^\eta$-proper for all $\mathbf{F}_1, \mathbf{F}_\eta \in \mathbb{H}^{n\times n}$ resulting in an invertible transformation $\bar{\mathbf{y}} = \overline{\mathbf{F}}^H \bar{\mathbf{x}}$.

Now, we can easily introduce the invariances and maximal invariants of the three testing problems.

3.2.1 Maximal invariant for $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathbb{C}^\eta}$

Assume that we are given T i.i.d. realizations $\mathbf{x}[t]$ ($t = 0,\dots,T-1$) of a zero-mean quaternion Gaussian vector $\mathbf{x}$, and define the augmented sample covariance matrix $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = \frac{1}{T}\sum_{t=0}^{T-1} \bar{\mathbf{x}}[t]\bar{\mathbf{x}}^H[t]$. Thus, it is easy to see that a sufficient statistic for the problem of testing $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathbb{C}^\eta}$ is

$$\hat{\mathbf{D}}_{\mathbb{C}^\eta} = \begin{bmatrix} \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}} & \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix}.$$

Moreover, noting that the testing problem is invariant under invertible *conventional* linear transformations, we can introduce a transformation $\mathbf{y}[t] = \mathbf{F}_1^H\mathbf{x}[t]$ such that $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}} = \mathbf{I}_n$ and $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta)}}$ is a real diagonal matrix, where the entries in the diagonal are given by the (ordered) sample canonical correlations [4, 14] between the random vectors $\mathbf{x}$ and $\mathbf{x}^{(\eta)}$. Thus, the n sample canonical correlations constitute a maximal invariant (under the group of invertible *conventional* linear transformations) for testing $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathbb{C}^\eta}$.

Finally, it is straightforward to prove that there exists a one-to-one correspondence between the n sample canonical correlations and the eigenvalues of the sample $\mathbb{R}^\eta$-coherence matrix [4], which is defined as $\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta} = \hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}\hat{\mathbf{D}}_{\mathbb{C}^\eta}\hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}$, with

$$\hat{\mathbf{D}}_{\mathbb{Q}} = \begin{bmatrix} \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix}.$$

3.2.2 Maximal invariant for $\mathscr{H}_{\mathbb{C}^\eta}$ versus $\mathscr{H}_{\mathscr{I}}$

In this case, the sufficient statistic is $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$, but taking into account the invariance of the testing problem under invertible *semi-widely* linear transformations, we can introduce a transformation $\mathbf{y}[t] = \mathbf{F}_1^H\mathbf{x}[t] + \mathbf{F}_\eta^H\mathbf{x}^{(\eta)}[t]$ such that $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}} = \mathbf{I}_n$, $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta)}} = \mathbf{0}_{n\times n}$, $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta')}} = \hat{\mathbf{\Sigma}}_{\eta'}$ and $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta'')}} = \hat{\mathbf{\Sigma}}_{\eta''}$, where

$$\hat{\mathbf{\Sigma}}_{\eta'} = \operatorname{diag}(\hat{\mathbf{c}}_1) - \operatorname{diag}(\hat{\mathbf{c}}_2), \quad \hat{\mathbf{\Sigma}}_{\eta''} = \operatorname{diag}(\hat{\mathbf{c}}_1) + \operatorname{diag}(\hat{\mathbf{c}}_2),$$

and $\mathbf{c} = \left[\mathbf{c}_1^T, \mathbf{c}_2^T\right]^T \in \mathbb{R}^{2n\times 1}$ are the (ordered) sample canonical correlations between the complex vectors $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$ and $\mathbf{a}^*$, i.e., $\mathbf{c}$ contains the sample circularity coefficients of the complex random vector $\mathbf{a}$ [8, 15]. Thus, as suggested by Lemma 4, the problem of testing for the $\mathbb{C}^\eta$-properness of $\mathbf{x}$ reduces to that of testing for the complex properness of $\mathbf{a}$, and the maximal invariant is given by the circularity coefficients $\mathbf{c}$, or equivalently, by the diagonal matrices $\hat{\mathbf{\Sigma}}_{\eta'}, \hat{\mathbf{\Sigma}}_{\eta''}$.

Finally, we must also point out that there exists a one-to-one correspondence between the sample circularity coefficients of $\mathbf{a}$ and the eigenvalues of the sample $\mathbb{C}^\eta$-coherence matrix [4], which is defined as $\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta} = \hat{\mathbf{D}}_{\mathbb{C}^\eta}^{-1/2}\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\hat{\mathbf{D}}_{\mathbb{C}^\eta}^{-1/2}$.

3.2.3 Maximal invariant for $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathscr{I}}$

This case is much more difficult than the previous ones. The testing problem is invariant under rotations and invertible *conventional* linear transformations, but the derivation of a maximal invariant is rather involved. Following the previous lines, we can see that the sufficient statistic $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$ can be written as

$$\begin{bmatrix} \widetilde{\mathbf{F}} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \widetilde{\mathbf{F}}^{(\eta')} \end{bmatrix}^{-H} \begin{bmatrix} \mathbf{I}_{2n} & \widetilde{\mathbf{\Sigma}} \\ \widetilde{\mathbf{\Sigma}} & \mathbf{I}_{2n} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{F}} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \widetilde{\mathbf{F}}^{(\eta')} \end{bmatrix}^{-1},$$

Table 1: GLRT statistics for Quaternion Gaussian Vectors

Test	GLRT statistic
$\mathscr{H}_{\mathbb{Q}}$ vs. $\mathscr{H}_{\mathscr{I}}$	$\hat{\mathscr{P}}_{\mathbb{Q}} = -\frac{1}{2}\ln\left\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\right\|$
$\mathscr{H}_{\mathbb{C}^\eta}$ vs. $\mathscr{H}_{\mathscr{I}}$	$\hat{\mathscr{P}}_{\mathbb{C}^\eta} = -\frac{1}{2}\ln\left\|\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}\right\|$
$\mathscr{H}_{\mathbb{Q}}$ vs. $\mathscr{H}_{\mathbb{C}^\eta}$	$\hat{\mathscr{P}}_{\mathbb{R}^\eta} = -\frac{1}{2}\ln\left\|\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}\right\|$

Table 2: LMPIT statistics for Quaternion Gaussian Vectors

Test	Invariances	LMPIT statistic
$\mathscr{H}_{\mathbb{Q}}$ vs. $\mathscr{H}_{\mathscr{I}}$	$\mathbf{F}_1^H\mathbf{x}^{(a)}$	$\left\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\right\|^2$
$\mathscr{H}_{\mathbb{C}^\eta}$ vs. $\mathscr{H}_{\mathscr{I}}$	$\mathbf{F}_1^H\mathbf{x}+\mathbf{F}_\eta^H\mathbf{x}^{(\eta)}$	$\left\|\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}\right\|^2$
$\mathscr{H}_{\mathbb{Q}}$ vs. $\mathscr{H}_{\mathbb{C}^\eta}$	$\mathbf{F}_1^H\mathbf{x}$	$\left\|\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}\right\|^2$

with

$$\widetilde{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_\eta^{(\eta)} \\ \mathbf{F}_\eta & \mathbf{F}_1^{(\eta)} \end{bmatrix}, \qquad \widetilde{\mathbf{\Sigma}} = \begin{bmatrix} \hat{\mathbf{\Sigma}}_{\eta'} & \hat{\mathbf{\Sigma}}_{\eta''} \\ \hat{\mathbf{\Sigma}}_{\eta''} & \hat{\mathbf{\Sigma}}_{\eta'} \end{bmatrix}.$$

Thus, introducing the transformation $\mathbf{y}[t] = \mathbf{F}_1^H\mathbf{x}[t]$ and defining $\mathbf{G} = \mathbf{F}_1^{-1}\mathbf{F}_\eta$, we can see that a maximal invariant (under invertible *conventional* linear transformations) for testing $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathscr{I}}$ is given by $\{\hat{\mathbf{\Sigma}}_{\eta'}, \hat{\mathbf{\Sigma}}_{\eta''}, \mathbf{G}\}$, i.e., $2n$ (ordered) sample canonical correlations and a quaternion matrix $\mathbf{G} \in \mathbb{H}^{n\times n}$, which is unambiguously specified up to individual products of its rows by unit quaternions in the plane $\{1,\eta\}$.

Here, it is obvious that the above maximal invariant does not have the simple form of those derived in the previous cases. Moreover, although intuitively appealing, there is not a one-to-one correspondence between the maximal invariant and the eigenvalues of the sample $\mathbb{Q}$-coherence matrix $\hat{\mathbf{\Phi}}_{\mathbb{Q}} = \hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}$ [4]. Finally, we should note that we have not yet considered the invariance of the test under quaternion rotations $\mathbf{y} = \mathbf{x}^{(a)}$. However, although this invariance could introduce a slight reduction in the degrees of freedom of the quaternion matrix $\mathbf{G}$, it does not seem to be enough for deriving a maximal invariant in the form of the previous cases.

3.3 Generalized Likelihood Ratio Tests (GLRTs)

The problem of testing for the properness of a quaternion Gaussian vector has been previously considered in [5, 6], where three generalized likelihood ratio tests (GLRTs) were derived. Table 1 shows the test statistics for our three testing problems, and the GLRTs reject the null (proper) hypothesis for high values of the GLRT statistics, which are always non-negative. Specifically, the test statistics $\hat{\mathscr{P}}_{\mathbb{Q}}, \hat{\mathscr{P}}_{\mathbb{C}^\eta}, \hat{\mathscr{P}}_{\mathbb{R}^\eta}$ can be seen as estimates of the quaternion improperness measures presented in [4], and they satisfy the relationship $\hat{\mathscr{P}}_{\mathbb{Q}} = \hat{\mathscr{P}}_{\mathbb{C}^\eta} + \hat{\mathscr{P}}_{\mathbb{R}^\eta}$, which has been used in [5] for introducing a multiple hypotheses test based on the three previous measures. Finally, the performance of the proposed GLRTs has been evaluated in [5,6] by means of simulations, whereas the complex counterpart, i.e., the performance analysis of the GLRT for testing the properness of complex random vectors has been addressed in [9, 16, 17].

3.4 Locally Most Powerful Invariant Test (LMPITs)

Although the GLRTs result in simple detectors and perform well in most practical situations, they can suffer from poor performance in the case of small sample sizes T. In order to solve this problem, in this subsection we present the locally most powerful invariant tests (LMPITs) for the three considered testing problems. As shown in Table 2, the test statistics are given by the Frobenius norm of the corresponding sample coherence matrices, and the LMPITs reject the null (proper) hypothesis for high values of the test statistic.

Due to the lack of space, we do not include here the (rather tedious) derivation of the test statistics, and refer the interested reader to the journal version of this paper [18]. However, we must point out that the derivation of the LMPITs follows the lines in [19] for the complex case. The key idea consists in using the Wijsman's theorem [7], which allows us to obtain the ratio between the maximal invariant densities, even without an explicit expression for the maximal invariant. With this idea in mind, a Taylor series expansion of the covariance matrices around their proper counterpart, and exploiting the invariances of each testing problem, one can finally come up with the expressions in Table 2.

4. FURTHER COMMENTS AND CONCLUSIONS

Let us here point out some important facts. Firstly, the GLRT and LMPIT statistics are functions of the eigenvalues of the corresponding coherence matrices, which was obvious in the cases of testing $\mathscr{H}_{\mathbb{C}^\eta}$ versus $\mathscr{H}_{\mathscr{I}}$, and $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathbb{C}^\eta}$, but it was not clear (although intuitively appealing) for the $\mathbb{Q}$-properness test. Secondly, it can be easily proved that the GLRT and LMPIT for testing the $\mathbb{C}^\eta$-properness of $\mathbf{x}$ reduce to the respective GLRT and LMPIT for testing the properness of the complex vector $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$ [8, 9]. Moreover, in the scalar case $x \in \mathbb{H}$ the proposed $\mathbb{Q}$-properness tests are equivalent to the GLRT [10] and LMPIT [11] for sphericity of the real vector $[r_1, r_\eta, r_{\eta'}, r_{\eta''}]^T$, whereas the tests for $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathbb{C}^\eta}$ are equivalent to the GLRT and LMPIT for sphericity of the proper complex vector $[a_1, a_2]^T$.

Regarding the practical performance of the proposed tests, we must note that the GLRTs preserve the invariances of the testing problem, which means that they will be outperformed by the LMPITs when the null (proper) and alternative (improper) hypotheses are *sufficiently close*. Here, we simply illustrate the performance of the tests by means of a numerical example with ten-dimensional ($n = 10$) quaternion Gaussian vectors with zero mean and SOS as specified in Table 3, where $\mathbf{\Lambda}_\eta$ is a diagonal matrix whose k-th diagonal entry is $\frac{10-k}{20}$, and $\mathbf{\Lambda}_{\eta'}$ is diagonal with entries $\frac{k-1}{40}$. The receiver operating characteristic (ROC) curves are shown in Figures 1 and 2, where we can see that, for practical sample sizes T, the proposed LMPITs outperform the GLRTs. Moreover, although the performance gap is moderate in the two simplest tests, the difference is very significative in the case of testing for $\mathbb{Q}$-properness (Fig. 2). Finally, future research will focus on a rigorous performance analysis, which should provide further insights on the tradeoffs among the number of vector observations, the *distance* between the hypotheses, and the practical performance of the tests.

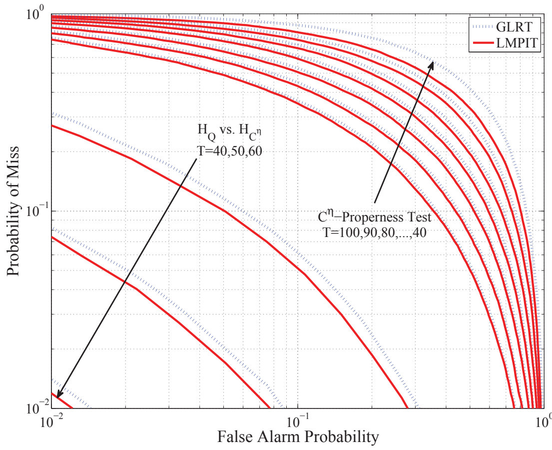

Figure 1: Receiver operating characteristic curves for the problems of testing $\mathscr{H}_{\mathbb{C}^\eta}$ versus $\mathscr{H}_{\mathscr{I}}$ ($\mathbb{C}^\eta$-properness test), and $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathbb{C}^\eta}$.

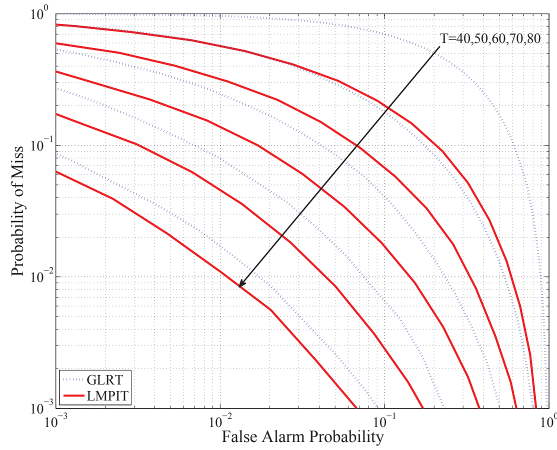

Figure 2: Receiver operating characteristic curves for the problem of testing $\mathscr{H}_{\mathbb{Q}}$ versus $\mathscr{H}_{\mathscr{I}}$ ($\mathbb{Q}$-properness test).

Table 3: Second Order Statistics for the Simulations

	$\mathbf{R}_{\mathbf{x},\mathbf{x}}$	$\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$	$\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$	$\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$
$\mathscr{H}_{\mathscr{I}}$	$\mathbf{I}_{10}$	$\mathbf{\Lambda}_\eta$	$\mathbf{\Lambda}_{\eta'}$	$\mathbf{0}_{10\times10}$
$\mathscr{H}_{\mathbb{C}^\eta}$	$\mathbf{I}_{10}$	$\mathbf{\Lambda}_\eta$	$\mathbf{0}_{10\times10}$	$\mathbf{0}_{10\times10}$
$\mathscr{H}_{\mathbb{Q}}$	$\mathbf{I}_{10}$	$\mathbf{0}_{10\times10}$	$\mathbf{0}_{10\times10}$	$\mathbf{0}_{10\times10}$

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